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Signatures of finite classical Lie algebra representations

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Abstract. The paper deals with the real classical Lie algebras of types B_r, C_r and their arbitrary irreducible representations. Hermitian forms invariant relative to these representations are considered. Signature formulas for these forms are obtained.

1. Introduction

Let $\mathfrak{g}$ be the simple complex classical Lie algebra and let $\mathfrak{g}_\sigma$ be any real form of inner type for $\mathfrak{g}$. Consider an irreducible representation $\varphi: \mathfrak{g} \rightarrow \mathfrak{sl}(V)$. From [1] it follows that $\varphi(\mathfrak{g}_\sigma) \subset \mathfrak{su}(p, q)$, where $p+q=\dim V$. Let $\delta=p-q$. So δ is a signature, i.e. the difference between the number of positive and negative signs in the bilinear invariant in its diagonal form. Furthermore $p=\frac{1}{2}(\dim V+\delta)$ and $q=\frac{1}{2}(\dim V-\delta)$. Hence it is possible to find the number of linearly independent space-like or time-like vectors in the representation space. In [1–5] formulas for δ were given in terms of the highest weight. If the rank of $\mathfrak{g}$ is small, then the formulas are rather simple [3–5]. As follows from this paper, it is possible to obtain simple δ formulas for some real forms $\mathfrak{g}_\sigma$ of arbitrary rank.

The finite-dimensional representations which are used in theoretical physics are mostly low-dimensional, nevertheless the interest to general methods is growing [5].

2. Definitions

Definitions used in this paper coincide with those in [4]. Let $\mathfrak{g}_\tau$ be the fixed compact real form of the algebra $\mathfrak{g}$ and let τ be the conjugation of algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_\tau$. Consider an involution θ of algebra $\mathfrak{g}$ such that $\theta(\mathfrak{g}_\tau)=\mathfrak{g}_\tau$. Let $\sigma=\tau \circ \theta=\theta \circ \tau$. Denote by $\mathfrak{g}_\sigma$ the real form of the algebra $\mathfrak{g}$ such that σ is a conjugation of algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_\sigma$. The real form $\mathfrak{g}_\sigma$ is called the real form of inner type if $\theta \in \mathrm{Int}(\mathfrak{g}_\tau)$. Suppose $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_\tau$ such that $\theta(\mathfrak{t})=\mathfrak{t}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}^{\mathbb{C}}=\mathfrak{h}$, R is a root system associated with the pair $(\mathfrak{g}, \mathfrak{h})$. Let $B(\ ,\ )$ be a Killing form of $\mathfrak{g}$, and let $(\ ,\ )=-\frac{1}{(2\pi)^2}B(\ ,\ )$ be a positive definite scalar product on $\mathfrak{t}$. Let $\alpha \in R$, by H_α denote an element of $\mathfrak{h}$ such that $B(H_\alpha, H)=\alpha(H)$ for all $H \in \mathfrak{h}$. Define the embedding $\psi: R \rightarrow \mathfrak{t}$ by $\psi(\alpha)=2\pi\sqrt{-1}H_\alpha$ for all $\alpha \in R$. Suppose $\Pi=\{\alpha_1, \ldots, \alpha_r\}$ is a set of the simple roots of R, $\{H_i\}_{i=1}^r$ is a basis of $\mathfrak{t}$ such that $(H_i, \alpha_j)=\delta_{ij}$, $i, j=1, \ldots, r$. If $\theta \in \mathrm{Int}(\mathfrak{g}_\tau)$, then without loss of generality $\theta=\exp(\mathrm{ad}(H_{i_0}/2))$ for some i_0, $1 \leqslant i_0 \leqslant r$ [6]. Let $R^\vee$ be the root system dual to R, that is, $R^\vee=\{2\alpha/(\alpha, \alpha) \mid \alpha \in R\}$.

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Suppose W is a Weyl group of R and $P(R^\vee)$ is a group of weights for $R^\vee$ [7], where $P(R^\vee)$ is generated by the elements $\{H_i\}_{i=1}^r$ mentioned above. Let λ be the highest weight of the representation $\varphi: \mathfrak{g} \to \mathfrak{sl}(V)$ and let χ_λ be the character of the representation φ. According to the Weyl character formula we have $A_\rho(H)\chi_\lambda(H) = A_{\lambda+\rho}(H)$, where $A_{\lambda+\rho}(H) = \Sigma_{s\in W} \det s \exp(2\pi\sqrt{-1}(s(\lambda+\rho), H))$ and $\rho = \frac{1}{2}\Sigma_{\beta\in R, \beta>0}\beta$ is half the sum of the positive roots R. Then [6]

$$A_\rho(H) = (2\sqrt{-1})^c \prod_{\beta\in R, \beta>0} \sin(\pi(\beta, H)) \tag{1}$$

where i is the number of positive roots. Denote by ω_i, $i = 1, \ldots, \operatorname{rank}(\mathfrak{g})$ basis representations of the algebra $\mathfrak{g}$, that is, $2(\omega_i, \alpha_k)/(\alpha_k, \alpha_k) = \delta_{ik}$, where $\alpha_k \in \Pi$, $i, k = 1, \ldots, \operatorname{rank}(\mathfrak{g})$. In accordance with [4] we shall call elements H_1 and $H_2 \in \mathfrak{h}$ equivalent if there exists $s \in W$ such that $s(H_1) - H_2 \in P(R^\vee)$ and we shall write $H \equiv H_2 (\operatorname{mod} P(R^\vee))$

Lemma 1. [4].
Let $\mathfrak{g}_\sigma$ be a real form of simple complex algebra $\mathfrak{g}$, $\theta = \sigma \circ \tau = \exp(\operatorname{ad}(H_{i_0}/2))$ and χ_λ be a character of the irreducible representation $\varphi: \mathfrak{g} \to \mathfrak{sl}(V)$. Then

$$|\delta| = |\chi_\lambda(H)| = \left| \lim_{t\to 1} \frac{A_{\lambda+\rho}(tH)}{A_\rho(tH)} \right| \tag{2}$$

where $H \equiv H_{i_0}/2 \ (\operatorname{mod} P(R^\vee))$.

3. The case $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$

The Dynkin diagram for $\mathfrak{so}(2r+1, \mathbb{C})$ is

$$\begin{array}{ccccccc} \alpha_1 & & \alpha_2 & & \alpha_{r-1} & & \alpha_r \\ 0 & — & 0 & — \ldots — & 0 & \Rightarrow & 0 \end{array}.$$

We shall take the roots realization from [7], that is,

$$\vec{\alpha}_i = \vec{\varepsilon}_i - \vec{\varepsilon}_{i+1} \qquad i = 1, \ldots, r-1 \qquad \vec{\alpha}_r = \vec{\varepsilon}_r. \tag{3}$$

Denote by R^+ the set of all positive roots relative to $\vec{\alpha}_1, \ldots, \vec{\alpha}_r$. Everywhere below in this paper sign '$\ldots \simeq \ldots t \to 1$' means that we keep only the lowest-degree term when $t \to 1$ for a function considered.

Lemma 2.
Let $\lambda = \Sigma_{j=1}^r \lambda_j \omega_j$ be a highest weight of the representation $\varphi: \mathfrak{so}(2r+1, \mathbb{C}) \to \mathfrak{sl}(V)$. Then

$$|A_\rho(t\rho)| \simeq (\pi(t-1))^{r(r-1)} 2^{r^2} 2!4! \ldots (2r-4)!(2r-2)! \qquad \text{when } t \to 1.$$

Let λ_r be even. Then

$$|A_\rho(t(\lambda+\rho))| \simeq (\pi(t-1))^{r(r-1)} 2^{r^2} \prod_{\beta\in R^+, \beta=\beta^\vee} (\beta, \lambda+\rho) \qquad \text{when } t \to 1.$$

Let λ_r be odd. Then

$$|A_\rho(t(\lambda+\rho))| \simeq (\pi(t-1))^{r^2} 2^{r^2} \prod_{\beta\in R^+} (\beta, \lambda+\rho) \qquad \text{when } t \to 1.$$

Proof.
From (3) it follows that positive roots for B_r are $\bar{\varepsilon}_p \pm \bar{\varepsilon}_q$, where $p<q, p=1, \ldots, r-1$, and $\bar{\varepsilon}_p, p=1, \ldots, r$. Furthermore,

$$\bar{\varepsilon}_p - \bar{\varepsilon}_q = \sum_{m=p}^{q-1} \alpha_m^\vee \qquad \bar{\varepsilon}_p + \bar{\varepsilon}_q = \sum_{m=p}^{q-1} \alpha_m^\vee + \sum_{m=q}^{r-1} 2\alpha_m^\vee + \alpha_r^\vee \tag{4}$$

where $p<q, p=1, \ldots, r-1$ and $\bar{\varepsilon}_p = \Sigma_{m=p}^{r-1} \alpha_m^\vee + \frac{1}{2}\alpha_r^\vee, p=1, \ldots, r$. Since $\rho = \frac{1}{2}\Sigma_{\beta \in R^+} \beta = \Sigma_{j=1}^r \omega_j$, we have

$$|A_\rho(t\rho)| = 2^{r^2} \left| \prod_{\beta \in R^+} \sin(\pi t(\beta, \rho)) \right|$$

$$\simeq (\pi(t-1))^{r(r-1)} 2^{r^2} 2!4! \ldots (2r-4)!(2r-2)! \qquad \text{when } t \to 1.$$

Suppose λ_r is even. Then from (1) and (4) it follows that

$$|A_\rho(t(\lambda+\rho))| \simeq (\pi(t-1))^{r(r-1)} 2^{r^2} \prod_{\beta \in R^+, \beta=\beta^\vee} (\beta, \lambda+\rho) \qquad \text{when } t \to 1.$$

Suppose λ_r is odd. Similarly, from (1) and (4) we derive

$$|A_\rho(t(\lambda+\rho))| \simeq (\pi(t-1))^{r^2} 2^{r^2} \prod_{\beta \in R^+} (\beta, \lambda+\rho) \qquad \text{when } t \to 1.$$

Theorem 1.
Suppose $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$, where $r \geqslant 2$, $\mathfrak{g}_\sigma = \mathfrak{so}_{1,2r}$ and $\lambda = \Sigma_{j=1}^r \lambda_j \omega_j$ is a highest weight of the representation φ. If λ_r is odd, then $\delta = 0$. If λ_r is even, then

$$|\delta| = \frac{\Pi_{\beta = R^+, \beta=\beta^\vee} (\beta, \lambda+\rho)}{2!4! \ldots (2r-2)!} \tag{5}$$

where the product embraces all positive $\beta \in R$, such that $\beta = \beta^\vee = 2\beta/(\beta, \beta)$. That is, table 1 is derived from equation (5) for the calculation of $|\delta|$.

Proof.
The element $H = \frac{1}{2}H_r$ defines automorphism $\theta = \exp(\mathrm{ad}\, H)$. Then

$$H_r = \frac{2\omega_r}{(\alpha_r, \alpha_r)} = 2\omega_r \qquad H_1 = \frac{2\omega_1}{(\alpha_1, \alpha_1)} = \omega_1 \qquad H_2 = \omega_2, \ldots, H_{r-1} = \omega_{r-1}.$$

Furthermore, $\frac{1}{2}H_r = \omega_r \equiv \omega_r + H_1 + \ldots + H_{r-1} = \rho (\mathrm{mod}\, P(R^\vee))$.

Table 1. Signature $|\delta|$ of the representation $\overset{\lambda_1}{0}$—$\overset{\lambda_2}{0}$— … —$\overset{\lambda_{r-1}}{0}$⇒$\overset{\lambda_r}{0}$ of $\mathfrak{so}_{1,2r}$.

λ_1	λ_2		λ_{r-1}	λ_r	$\lvert\delta\rvert$
0—	0—	…	—0⇒	0	
a	a		a	o	
0—	0—	…	—0⇒	0	0
a	a		a	e	
0—	0—	…	—0⇒	0	$\frac{\Pi_{\beta \in R^+, \beta=\beta^\vee} (\beta, \lambda+\rho)}{2!4! \ldots (2r-2)!}$

Symbol 'e'('o') in the column λ_r denotes an even (odd) λ_i. Symbol 'a' denotes any λ_i independent of whether it is even or odd.

Suppose λ_r is even, then from (2) it follows that

$$|\delta| = \left| \lim_{t \to 1} \frac{A_{\lambda+\rho}(t\rho)}{A_\rho(t\rho)} \right| = \left| \lim_{t \to 1} \frac{A_\rho(t(\lambda+\rho))}{A_\rho(t\rho)} \right|. \tag{6}$$

Also, formula (5) is derived from lemma 2.

Suppose λ_r is odd, then from lemma 2 and formula (6) it follows immediately that $\delta = 0$.

Lemma 3. [8].
Let $\lambda = \Sigma_{j=1}^r \lambda_j \omega_j$ be a highest weight of the representation $\varphi: \mathfrak{so}(2r+1, \mathbb{C}) \to \mathfrak{sl}(V)$. Consider a vector

$$\lambda + \rho = \sum_{j=1}^{r} (\lambda_j + 1)\omega_j = \sum_{q=1}^{r} h_q \bar{\varepsilon}_q$$

where $h_q = \frac{1}{2}(2\lambda_q + 2\lambda_{q+1} + \ldots + 2\lambda_{r-1} + \lambda_r + 2(r-q) + 1)$, $q = 1, \ldots, r-1$ and $h_r = \frac{1}{2}(\lambda_r + 1)$. For all $H \in \mathfrak{t}$ such that $H = \Sigma_{p=1}^r a_p \bar{\varepsilon}_p$ we have

$$A_{\lambda+\rho}(H) = 2^r \det[\sin(2\pi h_q a_p)] \tag{7}$$

where $\det[\sin(2\pi h_q a_p)]$ denotes the $r \times r$ determinant whose qp element is $\sin(2\pi h_q a_p)$.

Lemma 4.
Suppose $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$, where $r \geqslant 2$, $\mathfrak{g}_\sigma = \mathfrak{so}_{r-k, r+k+1}$, where $k = 0, 1, \ldots, r-2$. Let $H \in \mathfrak{t}$ define automorphism $\theta = \exp(\operatorname{ad} H)$ in the case of algebra $\mathfrak{g}_\sigma$. If $r-k$ is even, then $H = \frac{1}{2}H_i$, where $i = \frac{1}{2}(r-k)$. If $r+k+1$ is even, then $H = \frac{1}{2}H_i$, where $i = \frac{1}{2}(r+k+1)$.

(i) Let $m(k)$ be the number of positive roots $\beta \in R$ such that $(\beta, \frac{1}{2}H_i) \in \mathbb{Z}$. In other words $m(k) = \operatorname{card}\{\beta | \beta \in R^+, (\beta, \frac{1}{2}H_i) \in \mathbb{Z}\}$. Then $m(k) = \frac{1}{2}(r^2 - r + k^2 + k)$. The point $k = 0$ is the point of minimum value for the function $m: \mathbb{Z} \to \mathbb{Z}$. If $k > 0$, then the function $m = m(k)$ increases.

(ii) Consider a vector $\frac{1}{2}(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2(r-i))$ defined by its components in the basis $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$. Then

$$\tfrac{1}{2}H_i \equiv \tfrac{1}{2}(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2(r-i)) \qquad (\operatorname{mod} P(R^\vee)).$$

Furthermore

$$A_\rho(\tfrac{1}{2}t(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2(r-i)))$$
$$\simeq (\pi(t-1))^{m(k)} 2^{r^2} (0!2! \ldots (2i-2)!)(1!3! \ldots (2r-2i-1)!).$$

Proof.

(i) Note that $\frac{1}{2}H_i = \frac{1}{2}\omega_i = \frac{1}{2}\Sigma_{j=1}^i \bar{\varepsilon}_j$, $i = 1, 2, \ldots, r-1$. That is why $(\beta, \frac{1}{2}H_i) \in \mathbb{Z}$ if and only if $\beta = \bar{\varepsilon}_j, j = i+1, \ldots, r$, or $\beta = \bar{\varepsilon}_p \pm \bar{\varepsilon}_q$, where $p < q, p = i+1, \ldots, r-1$, or $\beta = \bar{\varepsilon}_p \pm \bar{\varepsilon}_q$, where $p < q, q = 2, \ldots, i$. Hence $m(k) = i^2 - i + (r-i)^2$. But $i = \frac{1}{2}(r-k)$ or $i = \frac{1}{2}(r+k+1)$. Therefore inserting i into $m(k)$ we have $m(k) = \frac{1}{2}(r^2 - r + k^2 + k)$.

(ii) Let r be odd. Then

$$\tfrac{1}{2}H_i \equiv \tfrac{1}{2}H_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} H_j = \tfrac{1}{2}(2\rho + 2\omega_r - \omega_i)$$
$$= \tfrac{1}{2}(2r-1, 2r-3, \ldots, 2r-(2i-1), 2r-2i, \ldots, 4, 2)$$
$$\equiv \tfrac{1}{2}(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2r-2i) \qquad (\operatorname{mod} P(R^\vee)).$$

Let r be even. Then

$$\tfrac{1}{2}H_i \equiv \tfrac{1}{2}H_i + \sum_{j=1}^{r} H_j = \tfrac{1}{2}(2\rho + 2\omega_r + \omega_i)$$
$$= \tfrac{1}{2}(2r+1, 2r-1, \ldots, 2r-(2i-3), 2r-2i, \ldots, 4, 2)$$
$$\equiv \tfrac{1}{2}(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2r-2i) \qquad (\bmod P(R^{\vee})).$$

So the lemma is proved.

Theorem 2.

Suppose $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$, where $r \geqslant 2$, $\mathfrak{g}_\sigma = \mathfrak{so}_{r-k, r+k+1}$, where $k = 0, 1, \ldots, r-2$, $\lambda = \Sigma_{j=1}^{r} \lambda_j \omega_j$ is a highest weight of the representation φ, $\lambda + \rho = \Sigma_{j=1}^{r} (\lambda_j + 1)\omega_j = \Sigma_{q=1}^{r} h_q \bar{\varepsilon}_q$. Let $i = \tfrac{1}{2}(r-k)$ in the case where $r-k$ is even and let $i = \tfrac{1}{2}(r+k+1)$ in the case where $r+k+1$ is even.

If λ_r is odd, then $\delta = 0$.

If λ_r is even, then

$$|\delta| = \frac{2^{m(k)} |\det(\zeta_{pq})|}{2^{r(r-1)}(0!2!\ldots(2i-2)!)(1!3!\ldots(2r-2i-1)!)} \tag{8}$$

where $\det(\zeta_{pq})$ denotes an $r \times r$ determinant whose pq element is ζ_{pq} and

$$\zeta_{pq} = \sin(\pi h_q)(h_q)^{2(p-1)} \qquad p = 1, \ldots, i, q = 1, \ldots, r$$
$$\zeta_{pq} = (h_q)^{2(p-i)-1} \qquad p = i+1, \ldots, r, q = 1, \ldots, r$$
$$h_q = \tfrac{1}{2}(2\lambda_q + 2\lambda_{q+1} + \ldots + 2\lambda_{r-1} + \lambda_r + 2(r-q) + 1) \qquad q = 1, \ldots, r-1$$
$$h_r = \tfrac{1}{2}(\lambda_r + 1).$$

Proof.

From lemmas 3 and 4 we derive

$$|\delta| = |\chi_\lambda(\tfrac{1}{2}(1, 3, \ldots, 2i-1, 2, 4, \ldots, 2r-2i))|$$
$$= \left| \lim_{t\to 1} \frac{2^r \det[\sin(t\pi(h_q a_p)]}{2^{r^2}(\pi(t-1))^{m(k)}(0!2!\ldots(2i-2)!)(1!3!\ldots(2r-2i-1)!)} \right|$$

where $a_p = 2p-1, p = 1, \ldots, i$, $a_p = 2p-2i, p = i+1, \ldots, r$. Furthermore

$$|\delta| = \left| \lim_{t\to 1} \frac{2^{m(k)} 2^r \Pi_{j=1}^{r} \sin(\pi t h_j) \det[\cos^{a_p-1}(\pi t h_q)]}{2^{r^2}(\pi(t-1))^{m(k)}(0!2!\ldots(2i-2)!)(1!3!\ldots(2r-2i-1)!)} \right|. \tag{9}$$

Let λ_r be odd. Making a Laplace expansion of the determinant in (9) by its first i columns we find that the lowest degree of $(t-1)$ in the numerator is $m(k)+i$. Hence $\delta = 0$.

Let λ_r be even. Discussing this in the same way and keeping only the lowest-degree terms when $t \to 1$ we derive formula (8) for the calculation of $|\delta|$. So ends the proof of the theorem.

A similar formula was found in [1].

Now we shall give the following definitions. Let $\lambda = \Sigma_{j=1}^{r} \lambda_j \omega_j$ be a highest weight of the representation φ, $\lambda + \rho = \Sigma_{j=1}^{r} (\lambda_j + 1)\omega_j = \Sigma_{q=1}^{r} h_q \bar{\varepsilon}_q$. A polynomial expression in z

$$f(z)=\prod_{j=1}^{r}(z-h_j\sin(\pi h_j))=\sum_{j=0}^{r}b_j(h)z^{r-j} \tag{10}$$

is associated with the representation φ. Consider a $(r-2)\times(r-2)$ matrix

$$F(\lambda)=\begin{matrix} b_{4-r} & b_{6-r} & \dots & b_{r-2} \\ b_{5-r} & b_{7-r} & \dots & b_{r-1} \\ \dots & \dots & & \dots \\ b_1 & b_3 & \dots & b_{2r-5} \end{matrix}$$

where $b_j, j=0,\dots,r$ are the coefficients in (10) and $b_j=0$ for $j<0$ and $j>r$. The matrix $F(\lambda)$ is associated with the representation φ. Consider the lower left-hand corner minors of the matrix $F(\lambda)$. Namely $G_0=1, G_1=b_1=\Sigma_{j=1}^{r} h_j\sin(\pi h_j)$, $G_2=b_0b_3-b_1b_2=\frac{1}{3}((\Sigma_{j=1}^{r} h_j\sin(\pi h_j))^3-\Sigma_{j=1}^{r} h_j^3\sin^3(\pi h_j)),\dots,G_{r-2}=\det(F(\lambda))$.The minor G_k is associated with the algebra $\mathfrak{so}_{r-k,r+k+1}$, $k=0,\dots,r-2$.

Theorem 3.
Suppose $\mathfrak{g}=\mathfrak{so}(2r+1,\mathbb{C})$, where $r\geqslant 2$, $\mathfrak{g}_\sigma=\mathfrak{so}_{r-k,r+k+1}$, where $k=0,1,\dots,r-2$, $\lambda=\Sigma_{j=1}^{r}\lambda_j\omega_j$ is a highest weight of the representation φ. Let $i=\frac{1}{2}(r-k)$ in the case where $r-k$ is even and let $i=\frac{1}{2}(r+k+1)$ in the case where $r+k+1$ is even. Suppose

$$X^\vee(\lambda)=\{\beta^\vee\mid\beta^\vee\in R^\vee,\beta^\vee>0,(\beta^\vee,\tfrac{1}{2}(\lambda+\rho))\in\mathbb{Z}\}\qquad C_\lambda^\vee=\prod_{\beta^\vee\in X^\vee(\lambda)}(\beta^\vee,\tfrac{1}{2}(\lambda+\rho)).$$

Let λ_r be even. Then $\operatorname{card}(X^\vee(\lambda))=\frac{1}{2}r(r-1)$ and

$$|\delta|=\frac{2^{\frac{1}{2}k(k+1)}|G_k|C_\lambda^\vee}{(0!2!\dots(2i-2)!)(1!3!\dots(2r-2i-1)!)} \tag{11}$$

where the minor G_k is associated with $\mathfrak{so}_{r-k,r+k+1}$.

Proof.
If λ_r is even, then $(\check{\varepsilon}_p,\frac{1}{2}(\lambda+\rho))\notin\mathbb{Z}, p=1,\dots,r$. If $(\check{\varepsilon}_p+\check{\varepsilon}_q,\frac{1}{2}(\lambda+\rho))\in\mathbb{Z}$, then $(\check{\varepsilon}_p-\check{\varepsilon}_q,\frac{1}{2}(\lambda+\rho))\notin\mathbb{Z}$ and *vice versa*. Hence $\operatorname{card}(X^\vee(\lambda))=\frac{1}{2}r(r-1)$. Formula (11) follows straightforwardly from formula (9).

Corollary 1.
If $\mathfrak{g}_\sigma=\mathfrak{so}_{r,r+1}$, then

$$|\delta|=\frac{C_\lambda^\vee}{0!1!\dots(r-1)!}. \tag{12}$$

If $\mathfrak{g}_\sigma=\mathfrak{so}_{r-1,r+2}$, then

$$|\delta|=\frac{2|\Sigma_{j=1}^{r}h_j\sin(\pi h_j)|C_\lambda^\vee}{(0!1!\dots(r-2)!)r!}. \tag{13}$$

If $\mathfrak{g}_\sigma=\mathfrak{so}_{r-2,r+3}$, then

$$|\delta|=\frac{\frac{8}{3}((\Sigma_{j=1}^{r}h_j\sin(\pi h_j))^3-\Sigma_{j=1}^{r}h_j^3\sin^3(\pi h_j))C_\lambda^\vee}{(0!1!\dots(r-3)!)(r-1)!(r+1)!} \tag{14}$$

where $h_q=\frac{1}{2}(2\lambda_q+2\lambda_{q+1}+\dots+2\lambda_{r-1}+\lambda_r+2(r-q)+1), q=1,\dots,r-1, h_r=\frac{1}{2}(\lambda_r+1)$.

Table 2. Signature δ of the representation $\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$ 0—0—0⇒0 of $\mathfrak{so}_{4,5}$, $\mathfrak{so}_{3,6}$.

λ_1 λ_2 λ_3 λ_4 0—0—0⇒0	Elements $C_\lambda^\vee$	$\lvert\delta\rvert$ for $\mathfrak{so}_{4,5}$	$\lvert\delta\rvert$ for $\mathfrak{so}_{3,6}$
a a a o 0—0—0⇒0	—	0	0
e e e e 0—0—0⇒0	$\frac{1}{64}A_2A_4A_6A_9A_{10}A_{12}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+\lambda_3+2)C_\lambda^\vee$
o e e e 0—0—0⇒0	$\frac{1}{64}A_1A_3A_6A_9A_{11}A_{12}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+2\lambda_2+\lambda_3+\lambda_4+5)C_\lambda^\vee$
e o e e 0—0—0⇒0	$\frac{1}{64}A_3A_5A_7A_9A_{10}A_{11}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}\lvert\lambda_1-\lambda_3\rvert C_\lambda^\vee$
e e o e 0—0—0⇒0	$\frac{1}{64}A_2A_3A_7A_8A_{10}A_{12}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+\lambda_3+\lambda_4+3)C_\lambda^\vee$
o o e e 0—0—0⇒0	$\frac{1}{64}A_1A_2A_4A_5A_7A_9$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+2\lambda_2+3\lambda_3+\lambda_4+7)C_\lambda^\vee$
o e o e 0—0—0⇒0	$\frac{1}{64}A_1A_4A_7A_8A_{11}A_{12}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+2\lambda_2+\lambda_3+4)C_\lambda^\vee$
e o o e 0—0—0⇒0	$\frac{1}{64}A_4A_5.A_6A_8A_{10}A_{11}$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}\lvert\lambda_1-\lambda_3-\lambda_4-1\rvert C_\lambda^\vee$
o o o e 0—0—0⇒0	$\frac{1}{64}A_1A_2A_3A_5A_6A_8$	$\frac{1}{12}C_\lambda^\vee$	$\frac{1}{24}(\lambda_1+2\lambda_2+3\lambda_3+2\lambda_4+8)C_\lambda^\vee$

Tables 2 and 3 are derived from (11)-(14) for the calculation of $|\delta|$ in the case $\mathfrak{g}=\mathfrak{so}(9,\mathbb{C})$. If $\mathfrak{g}=\mathfrak{so}(5,\mathbb{C})$ and $\mathfrak{g}=\mathfrak{so}(7,\mathbb{C})$, then the results are similar to that found in [4].

4. The case $\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$

The Dynkin diagram for $\mathfrak{sp}(2r,\mathbb{C})$ is

$$\begin{array}{l}\alpha_1\quad \alpha_2 \qquad\qquad \alpha_{r-1}\quad \alpha_r\\ 0\text{—}0\text{—}\ldots\text{—}0\Leftarrow 0\end{array}.$$

We shall take the roots realization from [7], that is,

$$\vec{\alpha}_i=\vec{\varepsilon}_i-\vec{\varepsilon}_{i+1} \qquad i=1,\ldots,r-1 \qquad \vec{\alpha}_r=2\vec{\varepsilon}_r. \tag{15}$$

Lemma 5.
Suppose $\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$, where $r\geqslant 3$, $\lambda=\Sigma_{j=1}^{r}\lambda_j\omega_j$ is a highest weight of the representation $\varphi:\mathfrak{sp}(2r,\mathbb{C})\to\mathfrak{sl}(V)$. Let $\lambda+\rho=\Sigma_{j=1}^{r}(\lambda_j+1)\omega_j=\Sigma_{q=1}^{r}h_q\vec{\varepsilon}_q$, where $h_q=\lambda_q+\lambda_{q+1}+\ldots+\lambda_r+r-q+1$, $q=1,\ldots,r$ and let $X^\vee(\lambda)=\{\beta^\vee|\beta^\vee\in R^\vee,\ \beta^\vee>0,\ (\beta^\vee,\frac{1}{2}(\lambda+\rho))\in\mathbb{Z}\}$. Then

$$\operatorname{card}(X^\vee(\lambda))=\frac{1}{2}\left(r^2-r+\left(\sum_{q=1}^{r}\cos(\pi h_q)\right)^2+\sum_{q=1}^{r}\cos(\pi h_q)\right)=m\left(\sum_{q=1}^{r}\cos(\pi h_q)\right) \tag{16}$$

where the function $m=m(k)$ is defined in lemma 4.

Table 3. Signature δ of the representation $\overset{\lambda_1}{0}$—$\overset{\lambda_2}{0}$—$\overset{\lambda_3}{0}$⇒$\overset{\lambda_4}{0}$ of $\mathfrak{so}_{2,7}$.

λ_1 λ_2 λ_3 λ_4 0—0—0⇒0	$\lvert\delta\rvert$ for $\mathfrak{so}_{2,7}$
a a a o 0—0—0⇒0	0
e e e e 0—0—0⇒0	$\frac{1}{270}\lvert(-h_1+h_2-h_3+h_4)^3-(-h_1^3+h_2^3-h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
o e e e 0—0—0⇒0	$\frac{1}{270}\lvert(h_1+h_2-h_3+h_4)^3-(h_1^3+h_2^3-h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
e o e e 0—0—0⇒0	$\frac{1}{270}\lvert(h_1-h_2-h_3+h_4)^3-(h_1^3-h_2^3-h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
e e o e 0—0—0⇒0	$\frac{1}{270}\lvert(h_1-h_2+h_3+h_4)^3-(h_1^3-h_2^3+h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
o o e e 0—0—0⇒0	$\frac{1}{270}\lvert(-h_1-h_2-h_3+h_4)^3-(-h_1^3-h_2^3-h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
o e o e 0—0—0⇒0	$\frac{1}{270}\lvert(-h_1-h_2+h_3+h_4)^3-(-h_1^3-h_2^3+h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
e o o e 0—0—0⇒0	$\frac{1}{270}\lvert(-h_1+h_2+h_3+h_4)^3-(-h_1^3+h_2^3+h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$
o o o e 0—0—0⇒0	$\frac{1}{270}\lvert(h_1+h_2+h_3+h_4)^3-(h_1^3+h_2^3+h_3^3+h_4^3)\rvert C^{\vee}_{\lambda}$

The elements $A_i, i=1,\ldots,12$ and the elements $h_i, i=1,\ldots,4$ must be taken from table 4.

Proof.
The value of $\mathrm{card}(X^{\vee}(\lambda))$ depends on whether $h_1,\ldots,h_r$ are even or odd. Let p be the number of odd integers among $h_1,\ldots,h_r$. Then

$$\mathrm{card}(X^{\vee}(\lambda))=p^2-p+(r-p)^2. \tag{17}$$

On the other hand $\Sigma_{q=1}^{r}\cos(\pi h_q)=\Sigma_{q=1}^{r}(-1)^{h_q}=r-2p$. That is, $p=\frac{1}{2}(r-\Sigma_{q=1}^{r}\cos(\pi h_q))$. Therefore inserting p into (17) we derive the desired result.

Theorem 4.
Suppose $\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$, where $r\geqslant 3$, $\mathfrak{g}_\sigma=\mathfrak{sp}_{2r}(\mathbb{R})$, $\lambda=\Sigma_{j=1}^{r}\lambda_j\omega_j$ is a highest weight of the representation $\varphi:\mathfrak{sp}(2r,\mathbb{C})\to\mathfrak{sl}(V)$. Let $\lambda+\rho=\Sigma_{q=1}^{r}h_q\vec{\varepsilon}_q$, where $h_q=\lambda_q+\lambda_{q+1}+\ldots+\lambda_r+r-q+1$, $q=1,\ldots,r$. If $m(\Sigma_{q=1}^{r}\cos(\pi h_q))=\frac{1}{2}(r(r-1))$, then

$$|\delta|=\frac{\Pi_{\beta\in X^{\vee}(\lambda)}(\beta^{\vee},\frac{1}{2}(\lambda+\rho))}{1!2!\ldots(r-1)!}. \tag{18}$$

If $m(\Sigma_{q=1}^{r}\cos(\pi h_q))>\frac{1}{2}(r(r-1))$, then $\delta=0$, where $m(k)=\frac{1}{2}(r^2-r+k^2+k)$ and $X^{\vee}(\lambda)$ is defined in lemma 5.

Proof.
The element $H=\frac{i}{2}H_r$ defines automorphism $\theta=\exp(\mathrm{ad}\,H)$. Then

$$H_r=\frac{2\omega_r}{(\alpha_r,\alpha_r)}=\frac{1}{2}\omega_r,\ H_1=\omega_1,\ldots,H_{r-1}=\omega_{r-1}.$$

Table 4. The elements A_i, $i=1,\ldots,12$ and the elements h_i, $i=1,\ldots,4$.

$A_1=\lambda_1+1$	$A_5=\lambda_2+1$	$A_9=\lambda_3+\lambda_4+2$
$A_2=\lambda_1+\lambda_2+2$	$A_6=\lambda_2+\lambda_3+2$	$A_{10}=\lambda_1+2\lambda_2+2\lambda_3+\lambda_4+6$
$A_3=\lambda_1+\lambda_2+\lambda_3+3$	$A_7=\lambda_2+\lambda_3+\lambda_4+3$	$A_{11}=\lambda_1+\lambda_2+2\lambda_3+\lambda_4+5$
$A_4=\lambda_1+\lambda_2+\lambda_3+\lambda_4+4$	$A_8=\lambda_3+1$	$A_{12}=\lambda_2+2\lambda_3+\lambda_4+4$
$h_1=\frac{1}{2}(2\lambda_1+2\lambda_2+2\lambda_3+\lambda_4+7)$	$h_2=\frac{1}{2}(2\lambda_2+2\lambda_3+\lambda_4+5)$	
$h_3=\frac{1}{2}(2\lambda_3+\lambda_4+3)$	$h_4=\frac{1}{2}(\lambda_4+1)$	

The elements A_i are the scalar products $(\beta^\vee,\lambda+\rho)$, where $\beta^\vee\in R^\vee$, $\beta^\vee>0$, $\beta=\beta^\vee$.

Furthermore,

$$\begin{aligned}\tfrac{1}{2}H_r&=\tfrac{1}{4}\omega_r\equiv\tfrac{1}{4}\omega_r+H_1+\ldots+H_{r-1}=\tfrac{1}{4}(\rho+3\omega_1+\ldots+3\omega_{r-1})\\&=\tfrac{1}{4}(\rho+\omega_1+\ldots+\omega_{r-1}+2(\omega_1+\ldots+\omega_{r-1}))\\&\equiv\tfrac{1}{4}(\rho+\omega_1+\ldots+\omega_{r-1})\qquad(\operatorname{mod}P(R^\vee)).\end{aligned}$$

Note that

$$\rho+\omega_1+\ldots+\omega_{r-1}=2\rho^\vee=\sum_{\beta^\vee\in R^\vee,\beta^\vee>0}\beta^\vee.$$

Hence from (3) it follows that

$$|\delta|=\left|\lim_{t\to1}\frac{A_{\lambda+\rho}(\frac{1}{2}t\rho^\vee)}{A_\rho(\frac{1}{2}t\rho^\vee)}\right|=\left|\lim_{t\to1}\frac{A_{\rho^\vee}(\frac{1}{2}t(\lambda+\rho))}{A_{\rho^\vee}(\frac{1}{2}t\rho)}\right|. \tag{19}$$

Discussing this as in lemma 4, we find

$$A_{\rho^\vee}(\tfrac{1}{2}t\rho)\simeq2r^2(\pi(t-1))^{\frac{1}{2}r(r-1)}(1!2!\ldots(r-1)!)\qquad\text{when } t\to1.$$

The necessary results follow immediately from lemma 5 and formulas (1) and (19).

Lemma 6.
Suppose $\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$, where $r\geqslant3$, $\mathfrak{g}_\sigma=\mathfrak{sp}_{i,r-i}$, where $i=1,2,\ldots,[r/2]$. Let $H\in\mathfrak{t}$ define automorphism $\theta=\exp(\operatorname{ad}H)$ in the case of algebra $\mathfrak{g}_\sigma$. Then $H=\frac{1}{2}H_i$. Let $c(i)$ be the number of positive roots $\beta\in R$ such that $(\beta,\frac{1}{2}H_i)\in\mathbb{Z}$. In other words $c(i)=\operatorname{card}\{\beta\mid\beta\in R^+,(\beta,\frac{1}{2}H_i)\in\mathbb{Z}\}$. Then $c(i)=i^2+(r-i)^2$. Consider a vector $\frac{1}{2}(1,3,\ldots,2r-2i-1,2,4,\ldots,2i)$ defined by its components in the basis $\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_r$. Then

$$\tfrac{1}{2}H_i\equiv\tfrac{1}{2}(1,3,\ldots,2r-2i-1,2,4,\ldots,2i)\qquad(\operatorname{mod}P(R^\vee)).$$

Furthermore

$$\begin{aligned}&A_\rho(\tfrac{1}{2}t(1,3,\ldots,2r-2i-1,2,4,\ldots,2i))\\&\qquad\simeq(\pi(t-1))^{c(i)}2^{r^2+i}(0!2!\ldots(2r-2i-2)!)\\&\qquad\quad\times(1!3!\ldots(2i-1)!)(1\cdot3\ldots(2r-2i-1))\qquad\text{when } t\to1.\end{aligned}$$

Proof.

$$\tfrac{1}{2}H_i \equiv \tfrac{1}{2}H_i + \sum_{\substack{j=1\\ j\neq i}}^{r} H_j = \tfrac{1}{2}(2\rho - \omega_i - \omega_r)$$

$$= \tfrac{1}{2}(2r-2, 2r-4, \ldots, 2r-2i, 2r-2i, 2r-2i-3, \ldots, 3, 1)$$

$$\equiv \tfrac{1}{2}(1, 3, \ldots, 2r-2i-1, 2, 4, \ldots, 2i) \qquad (\mathrm{mod}\ P(R^\vee)).$$

Discussing this as in lemma 4, we derive all necessary results.

Theorem 5.
Suppose $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{C})$, where $r \geqslant 3$, $\mathfrak{g}_\sigma = \mathfrak{sp}_{i,r-i}$, where $i = 1, 2, \ldots, [r/2]$, $\lambda = \Sigma_{j=1}^{r} \lambda_j \omega_j$ is a highest weight of the representation φ, $\lambda + \rho = \Sigma_{j=1}^{r} (\lambda_j + 1)\omega_j = \Sigma_{q=1}^{r} h_q \bar{\varepsilon}_q$. Then

$$|\delta| = \frac{|\det(\zeta_{pq})|}{2^{2i(r-i)} 0!2! \ldots (2r-2i-2)!1!3! \ldots (2i-1)!1 \cdot 3 \ldots (2r-2i-1)}. \tag{20}$$

where $\det(\zeta_{pq})$ denotes an $r \times r$ determinant whose pq element is ζ_{pq} and

$$\zeta_{pq} = \cos(\pi h_q)(h_q)^{2p-1} \qquad p = 1, \ldots, i, q = 1, \ldots, r$$

$$\zeta_{pq} = (h_q)^{2(p-i)-1} \qquad p = i+1, \ldots, r, q = 1, \ldots, r$$

$$h_q = \lambda_q + \lambda_{q+1} + \ldots + \lambda_r + r - q + 1 \qquad q = 1, \ldots, r.$$

Proof.
From root realizations for the algebras $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{C})$ it follows that their Weyl groups coincide. Hence it is possible to use formula (7) for the calculation $A_{\lambda+\rho}(H)$ in the case of $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{C})$. Furthermore, from lemma 6 we derive

$$|\delta| = |\chi_\lambda(\tfrac{1}{2}(1, 3, \ldots, 2r-2i-1, 2, 4, \ldots, 2i))|$$

$$= \left| \lim_{t\to 1} \frac{2^{2i(i-r)} \Pi_{j=1}^{r} \sin(\pi t h_j) \det[\cos^{a_p - 1}(\pi t h_q)]}{\pi(t-1)^{c(i)} 0!2! \ldots (2r-2i-2)!1!3! \ldots (2i-1)!1 \cdot 3 \ldots (2r-2i-1)} \right| \tag{21}$$

where $a_p = 2p$, $p = 1, \ldots, i$, and $a_p = 2p - 2i - 1$, $p = i+1, \ldots, r$. Discussing this as in theorem 2, we derive formula (20) for the calculation of δ. A similar formula was found in [1].

Lemma 7.
Let $\lambda = \Sigma_{j=1}^{r} \lambda_j \omega_j$ be a highest weight of the representation $\varphi: \mathfrak{sp}(2r, \mathbb{C}) \to \mathfrak{sl}(V)$ and let $\lambda + \rho = \Sigma_{j=1}^{r} (\lambda_j + 1)\omega_j = \Sigma_{q=1}^{r} h_q \bar{\varepsilon}_q$, where $h_q = \lambda_q + \lambda_{q+1} + \ldots + \lambda_r + r - q + 1$, $q = 1, \ldots, r$. Suppose $X(\lambda) = \{\beta | \beta \in R^+, (\beta, \tfrac{1}{2}(\lambda + \rho)) \in \mathbb{Z}\}$. Then

$$\mathrm{card}(X(\lambda)) = \tfrac{1}{2}\left(r^2 + \left(\sum_{q=1}^{r} \cos(\pi h_q)\right)^2\right).$$

Proof.
The proof is similar to that of lemma 5. Now we shall give the following definitions. Let $\lambda = \Sigma_{j=1}^{r} \lambda_j \omega_j$ be a highest weight of the representation $\varphi, \lambda + \rho =$

$\Sigma_{j=1}^{r}(\lambda_j+1)\omega_j=\Sigma_{q=1}^{r} h_q\bar{\varepsilon}_q$. A polynomial expression in z

$$f(z)=\prod_{j=1}^{r}(z-\cos(\pi t h_j))=\sum_{j=0}^{r} b_j(h,t)z^{r-j} \tag{22}$$

is associated with the representation φ. Consider the $(r-3)\times(r-3)$ matrix

$$F(\lambda)=\begin{matrix} b_{5-r} & b_{7-r} & \dots & b_{r-3} \\ \dots & \dots & & \dots \\ b_0 & b_2 & \dots & b_{2r-8} \\ b_1 & b_3 & \dots & b_{2r-7} \end{matrix}$$

where $b_j, j=0,\dots,r$ are the coefficients in (22) and $b_j=0$ for $j<0$ and $j>r$. The matrix $F(\lambda)$ is associated with the representation φ. Consider the lower left-hand corner minors of the matrix $F(\lambda)$. Namely $G_{-1}=G_0=1$, $G_1=b_1=\Sigma_{j=1}^{r}\cos(\pi t h_j)$, $G_2=b_0b_3-b_1b_2=\frac{1}{3}((\Sigma_{j=1}^{r}\cos(\pi t h_j))^3-\Sigma_{j=1}^{r}\cos^3(\pi t h_j)),\dots,G_{r-3}=\det(F(\lambda))$. The minor G_{r-2i-1} is called associated with the algebra $\mathfrak{sp}_{i,r-i}$, $i=1,2,\dots,[r/2]$.

Theorem 6.
Suppose $\mathfrak{g}=\mathfrak{sp}(2r,\mathbb{C})$, where $r\geqslant 3$, $\mathfrak{g}_\sigma=\mathfrak{sp}_{i,r-i}$, $i=1,2,\dots,[r/2]$, $\lambda=\Sigma_{j=1}^{r}\lambda_j\omega_j$ is a highest weight of the representation φ, $\lambda+\rho=\Sigma_{q=1}^{r}h_q\bar{\varepsilon}_q$. Suppose $X(\lambda)=\{\beta|\beta\in R^+,(\beta,\frac{1}{2}(\lambda+\rho))\in\mathbb{Z}\}$, $C_\lambda=\Pi_{\beta\in X(\lambda)}(\beta,\frac{1}{2}(\lambda+\rho))$. If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|>r-2i$, then $\delta=0$.

Proof.
Using formula (21) we find

$$|\delta|=\left|\lim_{t\to 1}\frac{2^{2i(t-r)}2^{\frac{1}{2}(r^2-r)}C_\lambda G_{r-2i-1}(\pi(t-1))^{\operatorname{card}X(\lambda)}}{\pi(t-1)^{c(i)}0!2!\dots(2r-2i-2)!1!3!\dots(2i-1)!1\cdot 3\dots(2r-2i-1)}\right|. \tag{23}$$

If card $X(\lambda)>c(i)$, then $\delta=0$. Hence the necessary result follows immediately from inequality

$$\tfrac{1}{2}\left(r^2+\left(\sum_{q=1}^{r}\cos(\pi h_q)\right)^2\right)>i^2+(r-i)^2.$$

Corollary.
(i) Let r be odd, $\mathfrak{g}_\sigma=\mathfrak{sp}_{\frac{1}{2}(r-1),\frac{1}{2}(r+1)}$. If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=1$, then

$$|\delta|=\frac{C_\lambda}{2^{\frac{1}{2}(r-1)}(1\cdot 3\dots r)(1!2!\dots(r-1)!)}. \tag{24}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|>1$, then $\delta=0$.
(ii) Let r be odd, $\mathfrak{g}_\sigma=\mathfrak{sp}_{\frac{1}{2}(r-3),\frac{1}{2}(r+3)}$.

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=3$, then

$$|\delta|=\frac{8C_\lambda}{2^{\frac{1}{2}(r-9)}(1\cdot 3\dots(r+2))(1!2!\dots(r-3)!)(r-1)!(r+1)!}. \tag{25}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=1$, then

$$|\delta|=\frac{\frac{1}{4}((\Sigma_{q=1}^{r}\cos(\pi h_q)h_q^2)^2-\Sigma_{q=1}^{r}\cos(\pi h_q)h_q^4)C_\lambda}{2^{\frac{1}{2}(r-9)}(1\cdot 3\ldots(r+2))(1!2!\ldots(r-3)!)(r-1)!(r+1)!}. \tag{26}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|>3$ then $|\delta|=0$.

(iii) Let r be even, $\mathfrak{g}_\sigma=\mathfrak{sp}_{\frac{1}{2}r,\frac{1}{2}r}$.

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=0$, then

$$|\delta|=\frac{C_\lambda}{2^{\frac{1}{2}r}(1\cdot 3\ldots(r-1))(1!2!\ldots(r-1)!)}. \tag{27}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|>0$, then $\delta=0$.

(iv) Let r be even, $\mathfrak{g}_\sigma=\mathfrak{sp}_{\frac{1}{2}(r-2),\frac{1}{2}(r+2)}$.

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=2$, then

$$|\delta|=\frac{2C_\lambda}{2^{\frac{1}{2}(r-4)}(1\cdot 3\ldots(r+1))(1!2!\ldots(r-2)!)r!}. \tag{28}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|=0$, then

$$|\delta|=\frac{\frac{1}{2}(\Sigma_{q=1}^{r}\cos(\pi h_q)h_q^2)C_\lambda}{2^{\frac{1}{2}(r-4)}(1\cdot 3\ldots(r+1))(1!2!\ldots(r-2)!)r!}. \tag{29}$$

If $|\Sigma_{j=1}^{r}\cos(\pi h_j)|>2$, then $|\delta|=0$.

Formulas (24)–(29) follow straightforwardly from formula (23). Tables 5 and 6 are derived from theorems 4 and 6 for the calculation of $|\delta|$ in the case $\mathfrak{g}=\mathfrak{sp}(8,\mathbb{C})$. If $\mathfrak{g}=$

Table 5. Signature δ of the representation λ_1 λ_2 λ_3 λ_4 0—0—0⇐0 of $\mathfrak{sp}_{2,2}$, $\mathfrak{sp}_{1,3}$.

λ_1 λ_2 λ_3 λ_4 0—0—0⇐0	Elements X_λ	$\lvert\delta\rvert$ for $\mathfrak{sp}_{2,2}$	$\lvert\delta\rvert$ for $\mathfrak{sp}_{1,3}$
o o o a 0—0—0⇐0	—	0	0
e e e a 0—0—0⇐0	$\frac{1}{16}R_2R_6R_7R_{11}$	$\frac{X_\lambda}{144}\prod_{j=1}^{4}h_j$	$\frac{X_\lambda}{720}\frac{1}{2}\lvert h_1^2-h_2^2+h_3^2-h_4^2\rvert\prod_{j=1}^{4}h_j$
o e e a 0—0—0⇐0	$\frac{1}{64}R_1R_3R_4R_6R_7R_{10}$	0	$\frac{X_\lambda}{720}2\prod_{j=1}^{4}h_j$
e o e a 0—0—0⇐0	$\frac{1}{16}R_3R_4R_5R_{12}$	$\frac{X_\lambda}{144}\prod_{j=1}^{4}h_j$	$\frac{X_\lambda}{720}\frac{1}{2}\lvert h_1^2-h_2^2-h_3^2+h_4^2\rvert\prod_{j=1}^{4}h_j$
e e o a 0—0—0⇐0	$\frac{1}{64}R_2R_3R_4R_8R_9R_{11}$	0	$\frac{X_\lambda}{720}2\prod_{j=1}^{4}h_j$
o o e a 0—0—0⇐0	$\frac{1}{64}R_1R_2R_5R_{10}R_{11}R_{12}$	0	$\frac{X_\lambda}{720}2\prod_{j=1}^{4}h_j$
o e o a 0—0—0⇐0	$\frac{1}{16}R_1R_8R_9R_{10}$	$\frac{X_\lambda}{144}\prod_{j=1}^{4}h_j$	$\frac{X_\lambda}{720}\frac{1}{2}\lvert h_1^2+h_2^2-h_3^2-h_4^2\rvert\prod_{j=1}^{4}h_j$
e o o a 0—0—0⇐0	$\frac{1}{64}R_5R_6R_7R_8R_9R_{12}$	0	$\frac{X_\lambda}{720}2\prod_{j=1}^{4}h_j$

Symbols e(o, a) have the same meaning in tables 1–6. The elements R_i, $i=1,\ldots,12$ and h_l, $l=1,\ldots,4$ must be taken from table 7.

Table 6. Signature δ of the representation $\overset{\lambda_1}{0}—\overset{\lambda_2}{0}—\overset{\lambda_3}{0}\Leftarrow\overset{\lambda_4}{0}$ of $sp_4(\mathbb{R})$.

$\overset{\lambda_1}{0}—\overset{\lambda_2}{0}—\overset{\lambda_3}{0}\Leftarrow\overset{\lambda_4}{0}$	$\lvert\delta\rvert$ for $sp_4(\mathbb{R})$
$\overset{e}{0}—\overset{e}{0}—\overset{e}{0}\Leftarrow\overset{e}{0}$	$\frac{1}{48}(\lambda_3+\lambda_4+2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+4)X_\lambda$
$\overset{e}{0}—\overset{e}{0}—\overset{e}{0}\Leftarrow\overset{o}{0}$	$\frac{1}{48}(\lambda_4+1)(\lambda_2+\lambda_3+\lambda_4+3)X_\lambda$
$\overset{e}{0}—\overset{o}{0}—\overset{e}{0}\Leftarrow\overset{e}{0}$	$\frac{1}{48}(\lambda_3+\lambda_4+2)(\lambda_2+\lambda_3+\lambda_4+3)X_\lambda$
$\overset{e}{0}—\overset{o}{0}—\overset{e}{0}\Leftarrow\overset{o}{0}$	$\frac{1}{48}(\lambda_4+1)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+4)X_\lambda$
$\overset{o}{0}—\overset{e}{0}—\overset{o}{0}\Leftarrow\overset{e}{0}$	$\frac{1}{48}(\lambda_2+\lambda_3+\lambda_4+3)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+4)X_\lambda$
$\overset{o}{0}—\overset{e}{0}—\overset{o}{0}\Leftarrow\overset{o}{0}$	$\frac{1}{48}(\lambda_4+1)(\lambda_3+\lambda_4+2)X_\lambda$
in other cases	0

Table 7. The elements R_i, $i=1,\ldots,12$ and the elements h_i, $i=1,\ldots,4$.

$R_1=\lambda_1+1$	$R_5=\lambda_2+1$	$R_9=\lambda_3+2\lambda_4+3$	
$R_2=\lambda_1+\lambda_2+2$	$R_6=\lambda_2+\lambda_3+2$	$R_{10}=\lambda_1+2\lambda_2+2\lambda_3+2\lambda_4+7$	
$R_3=\lambda_1+\lambda_2+\lambda_3+3$	$R_7=\lambda_2+\lambda_3+2\lambda_4+4$	$R_{11}=\lambda_1+\lambda_2+2\lambda_3+2\lambda_4+6$	
$R_4=\lambda_1+\lambda_2+\lambda_3+2\lambda_4+5$	$R_8=\lambda_3+1$	$R_{12}=\lambda_2+2\lambda_3+2\lambda_4+5$	
$h_1=\lambda_1+\lambda_2+\lambda_3+\lambda_4+4$	$h_2=\lambda_2+\lambda_3+\lambda_4+3$	$h_3=\lambda_3+\lambda_4+2$	$h_4=\lambda_4+1$

The elements R_i are the scalar products $(\beta, \lambda+\rho)$, where $\beta\in R^+$, $\beta=\beta^\vee$.

$sp(6, \mathbb{C})$, then the results are similar to that found in [4]. It is possible to calculate the signatures δ in the case of algebras $so_{2r}(\mathbb{C})$, $sl_{r+1}(\mathbb{C})$ by using the methods of this paper.

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